# Math 2270, Midterm 2

May 2nd, 2022

PRINT YOUR NAME: ______________________________

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 10 | |
| 10 | 10 | |
| 11 | 10 | |
| Total: | 110 | |

- No advanced calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit, except where otherwise indicated.
- You have 110 minutes and the exam is 110 points.
- You do not need to simplify numerical expressions. For example leave fractions like 100/7 or expressions like ln(3)/2 as is.
- Good luck!

1. (10 points) For each problem, fill in the blank with your answer or circle "Not enough info" if there is not enough information to answer the question.

   1. The linear system of equations $A\vec{x} = \vec{b}$ corresponds to the augmented matrix

   $$\left[\begin{array}{ccc|c} 2 & -2 & 4 & 0 \\ -3 & 0 & -15 & -6 \\ 3 & 0 & 15 & 6 \end{array}\right]$$

   with reduced row echelon form (RREF)

   $$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

   How many solutions does $A\vec{x} = \vec{b}$ have? Circle your answer.

   (i) 0 (ii) 1 (iii) $\infty$ (iv) Not enough info

   2. Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map defined by multiplying by the matrix $A$:

   $$A = \begin{bmatrix} -2 & 10 & 1 & 2 \\ -4 & 20 & 0 & 2 \\ 1 & -5 & 3 & -3 \end{bmatrix}.$$

   That is, $f(\vec{x}) = A\vec{x}$. The reduced row echelon form (RREF) of $A$ is

   $$\begin{bmatrix} 1 & -5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   Is $f$ onto, one-to-one, an isomorphism, or none of the above? Circle the best answer – if $f$ is an isomorphism, do not circle onto or one-to-one.

   (i) onto (ii) one-to-one (iii) isomorphism (iv) None of the above

   3. Which of the matrices is *NOT* invertible? Circle your answer.

   $(i)\ \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$ $(ii)\ \begin{bmatrix} -1 & -4 \\ 1 & 3 \end{bmatrix}$ $(iii)\ \begin{bmatrix} -2 & -5 \\ -1 & -3 \end{bmatrix}$ $(iv)\ \begin{bmatrix} 6 & -3 \\ 10 & -5 \end{bmatrix}$

4. What is the angle $\theta$ between the vectors $\vec{u}$ and $\vec{v}$?

$$\vec{u} = \begin{bmatrix} 3-\sqrt{15} \\ 0 \\ 123 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} -123 \\ 11 \\ 3-\sqrt{15} \end{bmatrix}.$$

$\theta =$ ______________ Not enough info

5. Consider the matrix

$$A = \begin{bmatrix} -9 & -4 \\ 6 & 1 \end{bmatrix}$$

as a matrix with coefficients in $\mathbb{F}_5 = \mathbb{Z}/5$. Is it invertible? Does there exist a matrix $A^{-1}$ with entries in $\mathbb{F}_5$ such that $A^{-1}A = I_2$ as matrices with coefficients in $\mathbb{F}_5$?

Invertible Not Invertible

2. (10 points) ☐ Skip

Solve the system of linear equations. Find the general solution in parametric form.

$$
\begin{aligned}
-x_1 \qquad\quad - 2x_3 &= -3 \\
x_1 \qquad\quad + 2x_3 &= 3 \\
-4x_1 + 5x_2 - 23x_3 &= 13
\end{aligned}
$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{\hspace{10cm}}$$

3. (10 points) ☐ Skip

Write

- $M_2(\mathbb{R})$ for the vector space of $2 \times 2$ matrices,
- $\mathbb{P}_2$ for the vector space of polynomials in a single variable $t$ with degree at most 2,
- $V$ for the vector space of continuous functions $g : \mathbb{R} \to \mathbb{R}$.

Which of the maps are linear? Circle your answer

1. $f : \mathbb{R}^3 \to \mathbb{R}$, $f(\vec{x}) = ||\vec{x}|| = \sqrt{\vec{x} \cdot \vec{x}}$.

   Linear     Not linear

2. $f : M_2(\mathbb{R}) \to \mathbb{R}$, $f(A) = \det A$.

   Linear     Not linear

3. Fix $B$ a $2 \times 2$ matrix. Define $f : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by $f(A) = AB - BA$.

   Linear     Not linear

4. Fix $p(t)$ in $\mathbb{P}_2$. Define $f : \mathbb{P}_2 \to \mathbb{P}_2$ by $f(q(t)) = 3q(t) - 2p'(t)$.

   Linear     Not linear

5. $f : V \to \mathbb{R}$ defined by $f(g) = g(0)$.

   Linear     Not linear

4. (10 points) ☐ Skip

Decide whether the linear system of equations $A\vec{x} = \vec{b}$ has a solution. If so, find the general solution.

If not, find the least squares solutions.

$$A = \begin{bmatrix} 4 & 0 \\ -2 & 0 \\ 2 & 1 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}$$

Does the system $A\vec{x} = \vec{b}$ have a solution?

Yes No

Least squares/Solution: ____________________

5. (10 points) ☐ Skip

Find a basis of solutions $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ to the Matrix differential equation:

$$\begin{aligned} x_1' &= 3x_1 + 2x_2 \\ x_2' &= -4x_1 - 3x_2 \end{aligned}$$

Basis of solutions: ______________________________

6. (10 points) ☐ Skip

Define a function $f : \mathbb{R}^3 \to \mathbb{R}$ that sends a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to the determinant of the matrix

$$\begin{bmatrix} a & 3 & 0 \\ b & -2 & -6 \\ c & 1 & 3 \end{bmatrix}.$$

This function is linear. Find the matrix $B$ for this function using the standard bases on $\mathbb{R}^3$ and $\mathbb{R}$.

$B = \begin{bmatrix} \_\_\_\_\_\_ & \_\_\_\_\_\_ & \_\_\_\_\_\_ \end{bmatrix}$

7. (10 points) ☐ Skip

Find the change of basis matrix $\underset{C\leftarrow B}{P}$ that rewrites a vector $[\vec{x}]_B$ in $B$ coordinates in terms of $C$ coordinates $[\vec{x}]_C$.

$$B = \left\{ \begin{bmatrix} -13 \\ -8 \end{bmatrix}, \quad \begin{bmatrix} 30 \\ 18 \end{bmatrix} \right\}$$

$$C = \left\{ \begin{bmatrix} -4 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} -5 \\ -4 \end{bmatrix} \right\}.$$

$$\underset{C\leftarrow B}{P} = \begin{bmatrix} \_\_\_\_\_\_ & \_\_\_\_\_\_ \\ \_\_\_\_\_\_ & \_\_\_\_\_\_ \end{bmatrix}$$

8. (10 points) ☐ Skip

Find a basis for the Null space and Column space for the matrix $A$.

$$A = \begin{bmatrix} -5 & -15 & 1 & 4 \\ 1 & 3 & -2 & -5 \\ -1 & -3 & -1 & 2 \end{bmatrix}$$

Basis for $\operatorname{Nul} A$: ____________________

Basis for $\operatorname{Col} A$: ____________________

9. (10 points) ☐ Skip

Use the Gram-Schmidt process to replace the given basis by an orthogonal basis.

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} \right\}$$

Orthogonal Basis: ______________________________

10. (10 points) ☐ Skip

Let $W$ be the subspace spanned by the vectors

$$\begin{bmatrix} 4 \\ -4 \\ -12 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -3 \\ -9 \\ -2 \end{bmatrix}.$$

Find a basis for the orthogonal complement $W^{\perp}$ of $W$.

Basis for $W^{\perp}$: ______________________________

11. (10 points) ☐ Skip

Find the inverse $A^{-1}$ of the matrix $A$:

$$A = \begin{bmatrix} 3 & -4 & 0 \\ -9 & 12 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\text{Inverse: } A^{-1} = \begin{bmatrix} \_\_\_\_\_\_ & \_\_\_\_\_\_ & \_\_\_\_\_\_ \\ \_\_\_\_\_\_ & \_\_\_\_\_\_ & \_\_\_\_\_\_ \\ \_\_\_\_\_\_ & \_\_\_\_\_\_ & \_\_\_\_\_\_ \end{bmatrix}$$