Worksheet 3A

Name: _____ Score: _____

Consider two bases, $\mathcal{B}$ and $\mathcal{C}$ for $\mathbb{R}^2$ or $\mathbb{R}^3$ and a vector $[\vec{x}]_{\mathcal{B}}$ in $\mathcal{B}$ coordinates. Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and rewrite the vector in $\mathcal{C}$ coordinates.

1.

$$\mathcal{C} = \left\{ \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right\} \qquad \mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ -12 \end{bmatrix} \right\}$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 13 \end{bmatrix}$$

2.

$$\mathcal{C} = \left\{ \begin{bmatrix} 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\} \qquad \mathcal{B} = \left\{ \begin{bmatrix} 4 \\ -9 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \end{bmatrix} \right\}$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -17 \\ 20 \end{bmatrix}$$

3.

$$\mathcal{C} = \left\{ \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -12 \\ 3 \end{bmatrix} \right\} \qquad \mathcal{B} = \left\{ \begin{bmatrix} 16 \\ -3 \end{bmatrix}, \begin{bmatrix} -16 \\ 3 \end{bmatrix} \right\}$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -23 \\ 14 \end{bmatrix}$$

4.

$$\mathcal{C} = \left\{ \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\} \qquad \mathcal{B} = \left\{ \begin{bmatrix} -6 \\ 9 \end{bmatrix}, \begin{bmatrix} 4 \\ -9 \end{bmatrix} \right\}$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -13 \\ 11 \end{bmatrix}$$

5.
$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\} \qquad \mathcal{C} = \left\{ \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 9 \end{bmatrix} \right\}$$
$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

6. Is it always true that
$$[c.\vec{x}]_{\mathcal{B}} = c\,[\vec{x}]_{\mathcal{B}}$$
and
$$[\vec{x} + \vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}?$$
That is, is the process of rewriting vectors in a new coordinate system $\mathcal{B}$ a linear map?

7. Find the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$ for two bases for the vector space $\mathbb{P}_2$ of polynomials of degree up to 2.

$$\mathcal{B} = \left\{x^2 + x + 1, x^2 + 1, x - 1\right\} \qquad \mathcal{C} = \left\{2x^2 + 3x + 1, 2x^2 + 2x + 1, -x^2 - 2\right\}$$

Use it to write the polynomial
$$p(x) = 1(x^2 + x + 1) + 2(x^2 + 1) + 3(x - 1)$$
in the new basis $\mathcal{C}$.

8. What are the columns of the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$? Hint: think of the matrix as the composite $P_{\mathcal{C}}^{-1}P_{\mathcal{B}}$. What are the columns of $P_{\mathcal{B}}$? What happens when you apply $P_{\mathcal{C}}^{-1}$ to them?

9. Suppose I want to convert from a basis $\mathcal{A}$ to a basis $\mathcal{C}$ and I already know the matrices
$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \qquad \underset{\mathcal{B}\leftarrow\mathcal{A}}{P}.$$
How do I find $\underset{\mathcal{C}\leftarrow\mathcal{A}}{P}$?

10. Which matrices are change-of-basis matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$? Are all matrices change-of-basis matrices?