Worksheet 4

Name: ____________________ Score: ________

1. Complex numbers $\mathbb{C}$ are $a + bi$, where a, b are real numbers $\mathbb{R}$. Practice multiplying:
   (a) $(4i)^3$
   (b) $(2 - 3i)(4 + 2i)$
   (c) $4i(8 - 12i)$

2. A complex number $a + bi$ corresponds to the point (a, b) in $\mathbb{R}^2$. Addition of complex numbers corresponds to vector addition in $\mathbb{R}^2$:

   $$(2 - 3i) + (4 + 12i) = 6 + 9i$$

   $$\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}.$$

   Ponder this momentarily.

   What about multiplication? This worksheet will describe what it corresponds to geometrically, i.e., in the plane $\mathbb{R}^2$.

3. The point of the complex numbers is to solve the equation $p(x) = x^2 + 1$. No matter what real number x you plug in, $p(x)$ is never zero. But i is a root: $i^2 = -1$, so $i^2 + 1 = 0$.

   Theorem: EVERY polynomial $p(x)$ has a root over the complex numbers. In fact, $p(x)$ factors into linear factors:

   $$p(x) = u \cdot (x - a_1)^{e_1}(x - a_2)^{e_2} \cdots (x - a_k)^{e_k}.$$

   Give three examples of polynomials that do not factor into linear factors over $\mathbb{R}$ but do over $\mathbb{C}$.

4. The *length* (or *magnitude*) of a vector $v = \begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$ is $||v|| = \sqrt{a^2 + b^2}$.

   Find the lengths of these vectors:
   (a) $(3, -2)$
   (b) $(-3, 4)$

   What is the length of a vector in $\mathbb{R}^3$?

5. Recall the formula

   $$e^{i\theta} = \cos\theta + i \sin\theta.$$

   Plot this point in $\mathbb{R}^2$ when $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}$. What is the length of the vector corresponding to $e^{i\theta}$? What is $e^{i\pi}$? What is $e^{i\pi} + 1$??

6. Any vector v in $\mathbb{R}^2$ can be rescaled to have length 1. What do you need to multiply by to make v have length 1?

   Write $\widehat{v}$ for the vector v, rescaled to have length 1. It points in the same direction as v but doesn't remember the length of v.

   Consider $\widehat{v}$ in $\mathbb{R}^2$ as being a complex number. How can you write $\widehat{v}$ in the form $e^{i\theta}$? What angle θ should you take?

   If $v = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, what is $\widehat{v}$ and what is the corresponding θ so that $\widehat{v} = e^{i\theta}$?

7. By definition, the original vector v is given by scaling out $\widehat{v}$ to have the right length:

$$v = ||v||\,\widehat{v}.$$

   Given any complex number $a + bi$, how can we write it as a real number multiplied by $e^{i\theta}$ for some θ? Use the above formula and rewrite $\widehat{v} = e^{i\theta}$.

   The angle θ such that $a + bi = re^{i\theta}$ is called the *angle* or *argument* of $a + bi$. The real number r is the length.

8. Suppose two complex numbers $a + bi$, $c + di$ can be written as

$$a + bi = r_1 e^{i\theta_1} \qquad c + di = r_2 e^{i\theta_2}.$$

   Take the product $(a+bi)(c+di)$ and rewrite in the form $re^{i\theta}$. What is its angle? What is its length?

   For example, if $a = 0, b = 1$, we are multiplying by i. What does this do, geometrically? Plot several points and their product with i.

   If $a + bi = e^{i\theta}$ for some θ, what does multiplying by $a + bi$ do to $c + di$ geometrically? Plot several examples for $\theta = \frac{\pi}{6}, \frac{\pi}{3}, \pi$.

9. Find the complex eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

   Diagonalize this matrix. Once you diagonalize, multiplying by this matrix is the same as multiplying each coordinate by the corresponding eigenvalue. Describe what multiplying by each of its eigenvalues looks like in $\mathbb{C} = \mathbb{R}^2$.

10. Why does every $n \times n$ matrix A over the complex numbers $\mathbb{C}$ have n eigenvalues, counted with multiplicity?