Worksheet ∞

Name: ____________________________ Score: __________

1 Topics

1. For each topic, explain: what is it? how do you find it or do it? what does it mean? What is its use and what does it do? Give examples.

   - row echelon form and reduced row echelon form ((R)REF)
   - How to solve a system of linear equations. When is a linear system of equations consistent or inconsistent?
   - linear combinations
   - the span $\text{Span}(v_1, \ldots, v_p)$ of a set of vectors (and how to find a basis for it)
   - when is a vector $\vec{b}$ in the span $\text{Span}\,(v_1, \ldots, v_p)$ of some other vectors?
   - When does a set of vectors $\{v_1, \ldots, v_p\}$ span all of $\mathbb{R}^n$? When is it linearly independent? How do you check and what does it mean?
   - linear functions
   - one to one/onto/isomorphism. How to check if a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is one of them?
   - The matrix of a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$. Give examples, such as rotation, projection, the identity matrix, etc. Draw pictures.
   - matrix multiplication, in particular multiplying a vector by a matrix and composition of linear functions
   - subspaces
   - bases
   - the null space, column space of a matrix A. (and how to find bases for them)
   - inverses (general formula and the 2x2 case). How to use A^{-1} to solve $A\vec{x} = \vec{b}$. When is a matrix invertible?
   - determinants (general formula and the 2x2 case)
   - Change of basis on $\mathbb{R}^n$.
   - General vector spaces V. Give examples: $\mathbb{P}_2, \mathbb{P}_n, \mathbb{P}$, continuous/differentiable functions $f : \mathbb{R} \to \mathbb{R}$, the vector space of all 3x2 matrices, etc.
   - the coordinates of a vector $\vec{v}$ in V in a basis B for V.
   - The matrix of a linear map $f : V \to W$ between general vector spaces. (need to choose bases for V and W) Give examples, such as the derivative, integral, or indefinite integral of the vector space of polynomials
   - eigenvalues and eigenvectors

- diagonalizing matrices
- Matrix differential equations $\vec{x}'(t) = A\vec{x}(t)$ and initial value problems where $\vec{x}(0)$ is given
- Complex numbers $\mathbb{C}$ and complex eigenvalues/vectors
- The dot product
- The length of a vector $\vec{u}$ and the angle between two vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$.
- orthogonal and orthonormal sets $\{v_1, \ldots, v_p\}$ of vectors
- The orthogonal complement $W^\perp$ of a subspace W of $\mathbb{R}^n$. How to find a basis for it? $(\mathrm{Col}\, A)^\perp = \mathrm{Nul}(A^T)$. (This and least squares were our only uses of the transpose)
- the Gram Schmidt process
- Least squares solutions to $A\vec{x} = \vec{b}$. (multiply by A^T: $A^T A\vec{x} = A^T\vec{b}$.)
- Anything I forgot?

Some problems:

2. Find a basis of solutions for the Matrix differential equation

$$\begin{aligned} x_1' &= 2x_1 - 2x_2 \\ x_2' &= 4x_1 - 4x_2 \end{aligned}$$

Eigenvalues and vectors:

3. Find the angle between the two vectors. Find the length of each vector.

- $$\begin{bmatrix} 13 \\ -2 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ 13 \end{bmatrix}$$
- $$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \begin{bmatrix} \sqrt{6}+\sqrt{2} \\ \sqrt{6}-\sqrt{2} \end{bmatrix}$$
- $$\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ -13 \\ 0 \end{bmatrix}$$

4. Is the vector $\vec{b}$ in the span of the other vectors?

- $$\vec{b} = \begin{bmatrix} -22 \\ 5 \\ -5 \end{bmatrix}$$

  $$\begin{bmatrix} -6 \\ 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 10 \\ 1 \\ -1 \end{bmatrix}$$

- $$\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

  $$\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ -5 \end{bmatrix} \quad \begin{bmatrix} -10 \\ 25 \\ -5 \end{bmatrix}$$

5. Find the determinants

   - $$\begin{bmatrix} 45 & 12 & 12 \\ -48 & -13 & -12 \\ -12 & -3 & -4 \end{bmatrix}$$

   - $$\begin{bmatrix} 6 & 38 & 46 \\ 0 & 2 & 4 \\ -1 & -8 & -11 \end{bmatrix}$$

   Determinant:

6. Find the inverses

   - $$\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$$

   - $$\begin{bmatrix} -4 & -3 \\ -1 & -1 \end{bmatrix}$$

   Inverse:

7. Are the vectors

   $$\begin{bmatrix} 1 \\ -1 \\ -3 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -7 \\ 3 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} 7 \\ -6 \\ 4 \\ -2 \end{bmatrix},$$

   linearly independent?

8. Do the vectors
$$\begin{bmatrix}-2\\-2\\-9\end{bmatrix}\quad\begin{bmatrix}0\\-2\\-1\end{bmatrix}\quad\begin{bmatrix}4\\4\\18\end{bmatrix}\quad\begin{bmatrix}-4\\-8\\-20\end{bmatrix}$$
span $\mathbb{R}^3$?

9. Find a basis for the null space and column space of the matrix
$$\begin{bmatrix}0 & -9 & 9 & 36\\-2 & 2 & -10 & -6\\5 & 4 & 16 & -21\end{bmatrix}$$

10. Find a basis for the span of the vectors:
$$\begin{bmatrix}4\\0\\-2\end{bmatrix}\quad\begin{bmatrix}0\\-2\\-5\end{bmatrix}\quad\begin{bmatrix}8\\2\\1\end{bmatrix}\quad\begin{bmatrix}-4\\3\\2\end{bmatrix}$$

11. Find the change of basis matrix to get from B to C:
$$B=\begin{bmatrix}-5\\1\end{bmatrix},\quad\begin{bmatrix}17\\1\end{bmatrix},$$
$$C=\begin{bmatrix}-5\\1\end{bmatrix},\quad\begin{bmatrix}-2\\-4\end{bmatrix},\quad .$$

12. Let W be the subspace spanned by the vectors
$$\begin{bmatrix}2\\0\\3\\-9\end{bmatrix},\quad\begin{bmatrix}-1\\0\\-3\\12\end{bmatrix}.$$
Find a basis for the orthogonal complement $W^{\perp}$ of W.

13. Find the eigenvalues and eigenvectors:
$$\begin{bmatrix}7 & -12\\2 & -3\end{bmatrix}$$

14. Diagonalize the matrix:
$$\begin{bmatrix}-5 & 9\\-6 & 10\end{bmatrix}$$